

# IMPROVING THE CONSTANTS FOR THE REAL AND COMPLEX BOHNENBLUST-HILLE INEQUALITY

DANIEL PELLEGRINO AND JUAN B. SEOANE-SEPÚLVEDA\*

ABSTRACT. A classical inequality due to Bohnenblust and Hille states that for every  $N \in \mathbb{N}$  and every  $m$ -linear mapping  $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \rightarrow \mathbb{C}$  we have

$$\left( \sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|U\|,$$

where  $C_m = 2^{\frac{m-1}{2}}$  (in fact a recent remark of A. Defant and P. Sevilla-Peris indicates that  $C_m \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ ). Bohnenblust-Hille inequality is also true for real Banach spaces with the constants  $C_m = 2^{\frac{m-1}{2}}$ . In this note we show that an adequate use of a recent new proof of Bohnenblust-Hille inequality, due to Defant, Popa and Schwarting, combined with the optimal constants of Khinchine's inequality (due to Haagerup) provides quite better estimates for the constants involved in both real and complex Bohnenblust-Hille inequalities. For instance, in the real case, for  $2 \leq m \leq 14$ , we show that the constants  $C_m = 2^{\frac{m-1}{2}}$  can be replaced by  $2^{\frac{m^2+6m-8}{8m}}$  if  $m$  is even and by  $2^{\frac{m^2+6m-7}{8m}}$  if  $m$  is odd, improving, in this way, the known values of  $C_m$ . In both complex and real cases, the new constants are asymptotically better.

## 1. PRELIMINARIES AND BACKGROUND

In 1931, Bohnenblust and Hille ([2], or the more recent [8, 9]) asserted that for every positive integer  $N$  and every  $m$ -linear mapping  $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \rightarrow \mathbb{C}$  we have

$$\left( \sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|U\|,$$

where  $C_m = 2^{\frac{m-1}{2}}$  (actually this result also holds for real Banach spaces). The case  $m = 2$  is a famous result known as Littlewood's 4/3-inequality. It seems that the Bohnenblust-Hille inequality was overlooked and was only re-discovered several decades later by Davie [6] and Kaijser [13]. While the exponent  $\frac{2m}{m+1}$  is optimal, the constant  $C_m = 2^{\frac{m-1}{2}}$  is not. Very recently, Defant and Sevilla-Peris [8, Section 4] indicated that by using Sawa's estimate for the constant of the complex Khinchine's inequality in Steinhaus variables (see [16]) it is possible to prove that  $C_m \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  in the complex case (this is a strong improvement on the previous constants and it seems that these are the best known estimates for the complex case).

The (complex and real) Bohnenblust-Hille inequality can be re-written in the context of multiple summing multilinear operators, as we will see next. Multiple summing multilinear mappings between Banach spaces is a recent, very important and useful nonlinear generalization of the concept of absolutely summing linear operators. This class was introduced, independently, by Matos [15] (under the terminology fully summing multilinear mappings) and Bombal, Pérez-García and Villanueva [3]. The interested reader can also refer to [5, 4] for other Bohnenblust-Hille type results.

Throughout this paper  $X_1, \dots, X_m$  and  $Y$  will stand for Banach spaces over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $X'$  stands for the dual of  $X$ . By  $\mathcal{L}(X_1, \dots, X_m; Y)$  we denote the Banach space of all continuous

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$m$ -linear mappings from  $X_1 \times \cdots \times X_m$  to  $Y$  with the usual sup norm. For  $x_1, \dots, x_n$  in  $X$ , let

$$\|(x_j)_{j=1}^n\|_{w,1} := \sup\{\|(\varphi(x_j))_{j=1}^n\|_1 : \varphi \in X', \|\varphi\| \leq 1\}.$$

If  $1 \leq p < \infty$ , an  $m$ -linear mapping  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(p; 1)$ -summing (denoted  $\Pi_{(p;1)}(X_1, \dots, X_m; Y)$ ) if there exists a constant  $L_m \geq 0$  such that

$$(1.1) \quad \left( \sum_{j_1, \dots, j_m=1}^N \|U(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)})\|^p \right)^{\frac{1}{p}} \leq L_m \prod_{k=1}^m \left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1}$$

for every  $N \in \mathbb{N}$  and any  $x_{j_k}^{(k)} \in X_k$ ,  $j_k = 1, \dots, N$ ,  $k = 1, \dots, m$ . The infimum of the constants satisfying (1.1) is denoted by  $\|U\|_{\Pi_{(p;1)}}$ . For  $m = 1$  we have the classical concept of absolutely  $(p; 1)$ -summing operators (see, e.g. [7, 10]).

A simple reformulation of Bohnenblust-Hille inequality asserts that every continuous  $m$ -linear form  $T : X_1 \times \cdots \times X_m \rightarrow \mathbb{K}$  is multiple  $(\frac{2m}{m+1}; 1)$ -summing with  $L_m = C_m = 2^{\frac{m-1}{2}}$  (or  $L_m = (\frac{2}{\sqrt{\pi}})^{m-1}$  for the complex case, using the estimates of Defant and Sevilla-Peris, [8]). However, in the real case the best constants known seem to be  $C_m = 2^{\frac{m-1}{2}}$ .

The main goal of this paper is to obtain *better* constants for the Bohnenblust-Hille inequality in the real and complex case. For this task we will use a recent proof of a general vector-valued version of Bohnenblust-Hille inequality ([9, Theorem 5.1]). The Bohnenblust-Hille inequality is stated in [9, Corollary 2] as a consequence of [9, Theorem 5.1]. The procedure of the proof of [9, Corollary 2] allows us to obtain much better values than  $C_m = 2^{\frac{m-1}{2}}$ . However, in this note we explore the ideas of [9] in a different way, in order to obtain even better estimates for the constants that can be derived from [9, Corollary 2]. The constants we obtain here can be derived from [9, Theorem 5.1] via an adequate choice of variables.

Let us recall some results that we will need in this note. The first result is a well-known inequality due to Khinchine (see [10]):

**Theorem 1.1** (Khinchine's inequality). *For all  $0 < p < \infty$ , there exist constants  $A_p$  and  $B_p$  such that*

$$(1.2) \quad A_p \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

for every positive integer  $N$  and scalars  $a_1, \dots, a_n$  (here,  $r_n$  denotes the  $n$ -Rademacher function).

Above, it is clear that  $B_2 = 1$ . From (1.2) it follows that

$$(1.3) \quad \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p A_r^{-1} \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^r dt \right)^{\frac{1}{r}}$$

and the product of the constants  $B_p A_r^{-1}$  will appear later on in Theorem 1.3. The notation  $A_p$  and  $B_p$  will be kept along the paper. Next, let us recall a variation of an inequality due to Blei (see [9, Lemma 3.1]).

**Theorem 1.2** (Blei, Defant et al.). *Let  $A$  and  $B$  be two finite non-void index sets, and  $(a_{ij})_{(i,j) \in A \times B}$  a scalar matrix with positive entries, and denote its columns by  $\alpha_j = (a_{ij})_{i \in A}$  and its rows by  $\beta_i = (a_{ij})_{j \in B}$ . Then, for  $q, s_1, s_2 \geq 1$  with  $q > \max(s_1, s_2)$  we have*

$$\left( \sum_{(i,j) \in A \times B} a_{ij}^{w(s_1, s_2)} \right)^{\frac{1}{w(s_1, s_2)}} \leq \left( \sum_{i \in A} \|\beta_i\|_q^{s_1} \right)^{\frac{f(s_1, s_2)}{s_1}} \left( \sum_{j \in B} \|\alpha_j\|_q^{s_2} \right)^{\frac{f(s_2, s_1)}{s_2}},$$

with

$$w : [1, q]^2 \rightarrow [0, \infty), \quad w(x, y) := \frac{q^2(x + y) - 2qxy}{q^2 - xy},$$

$$f : [1, q]^2 \rightarrow [0, \infty), \quad f(x, y) := \frac{q^2x - qxy}{q^2(x + y) - 2qxy}.$$

The following theorem is a particular case of [9, Lemma 2.2] for  $Y = \mathbb{K}$  using that the cotype 2 constant of  $\mathbb{K}$  is 1, i.e.,  $C_2(\mathbb{K}) = 1$  (following the notation from [9]):

**Theorem 1.3** (Defant et al). *Let  $1 \leq r \leq 2$ , and let  $(y_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^N$  be a matrix in  $\mathbb{K}$ . Then*

$$\left( \sum_{i_1, \dots, i_m=1}^N |y_{i_1 \dots i_m}|^2 \right)^{1/2} \leq (A_{2,r})^m \left( \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^N r_{i_1}(t_1) \dots r_{i_m}(t_m) y_{i_1 \dots i_m} \right|^r dt_1 \dots dt_m \right)^{1/r},$$

where

$$A_{2,r} \leq A_r^{-1} B_2 = A_r^{-1} \quad (\text{since } B_2 = 1).$$

The meaning of  $A_{2,r}$ ,  $w$  and  $f$  from the above theorems will also be kept in the next section and  $K_G$  will denote the complex Grothendieck constant.

## 2. IMPROVED CONSTANTS FOR THE BOHNENBLUST-HILLE THEOREM: THE REAL CASE

The main results from [9], Theorem 5.1 and Corollary 5.2, are very interesting vector-valued generalizations of the Bohnenblust-Hille inequality. In this note we explore the proof of [9, Theorem 5.1] in such a way that the constants obtained are better than those that can be derived from [9, Corollary 5.2]. We will use here a modification of the proof of [9, Corollary 5.2] for the particular case of the Bohnenblust-Hille inequality applying some changes, partly inspired by [9, Theorem 5.1], improving the constants. By doing this, we will avoid some technicalities from the arguments from [9, Theorem 5.1] that are not needed here.

As we said, following the proof of [9, Corollary 5.2] and using the optimal values for the constants of Khinchine's inequality (due to Haagerup), the following estimates can be calculated for  $C_m$ :

$$(2.1) \quad C_{\mathbb{R},2} = \sqrt{2},$$

$$(2.2) \quad C_{\mathbb{R},m} = 2^{\frac{m-1}{2m}} \left( \frac{C_{\mathbb{R},m-1}}{A_{\frac{2m-2}{m}}} \right)^{1-\frac{1}{m}} \quad \text{for } m \geq 3.$$

In particular, if  $2 \leq m \leq 13$ ,

$$(2.3) \quad C_{\mathbb{R},m} \leq 2^{\frac{m^2+m-2}{4m}}.$$

**Remark 2.1.** *It is worth mentioning that the above constants are not explicitly calculated in [9]. Since our procedure below will provide better constants, we will not give much detail on the above estimates.*

A different approach on some of the ideas from [9] can give better estimates for the real case, as we see in the following result.

**Theorem 2.2.** *For every positive integer  $m$  and real Banach spaces  $X_1, \dots, X_m$ ,*

$$\Pi_{(\frac{2m}{m+1}, 1)}(X_1, \dots, X_m; \mathbb{R}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{R}) \quad \text{and} \quad \|\cdot\|_{\Pi_{(\frac{2m}{m+1}, 1)}} \leq C_{\mathbb{R},m} \|\cdot\|$$

with

$$(2.4) \quad C_{\mathbb{R},2} = 2^{\frac{1}{2}} \quad \text{and} \quad C_{\mathbb{R},3} = 2^{\frac{5}{6}},$$

$$C_{\mathbb{R},m} \leq 2^{\frac{1}{2}} \left( \frac{C_{\mathbb{R},m-2}}{A_{\frac{2m-4}{m-1}}} \right)^{\frac{m-2}{m}} \quad \text{for } m > 3.$$

In particular, if  $2 \leq m \leq 14$ ,

$$(2.5) \quad C_{\mathbb{R},m} \leq 2^{\frac{m^2+6m-8}{8m}} \quad \text{if } m \text{ is even, and}$$

$$(2.6) \quad C_{\mathbb{R},m} \leq 2^{\frac{m^2+6m-7}{8m}} \text{ if } m \text{ is odd.}$$

*Proof.* The case  $m = 2$  is Littlewood's  $4/3$ -inequality. For  $m = 3$  we have  $C_{\mathbb{R},3} = 2^{\frac{5}{6}}$  from (2.3). We proceed by induction, but the case  $m$  is obtained as a combination of the cases 2 with  $m - 2$  instead of 1 and  $m - 1$  as in [9, Corollary 5.2].

Suppose that the result is true for  $m - 2$  and let us prove for  $m$ . Let  $U \in \mathcal{L}(X_1, \dots, X_m; \mathbb{R})$  and  $N$  be a positive integer. For each  $1 \leq k \leq m$  consider  $x_1^{(k)}, \dots, x_N^{(k)} \in X_k$  so that  $\left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1} \leq 1$ ,  $k = 1, \dots, m$ .

Consider, in the notation of Theorem 1.2,

$$q = 2, \quad s_1 = \frac{4}{3}, \quad \text{and} \quad s_2 = \frac{2(m-2)}{(m-2)+1} = \frac{2m-4}{m-1}.$$

Thus,

$$w(s_1, s_2) = \frac{2m}{m+1}$$

and, from Theorem 1.2, we have

$$\begin{aligned} & \left( \sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} \leq \\ & \leq \left( \sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_{m-1}, i_m=1}^N \right\|_2^{\frac{2(m-2)}{(m-2)+1}} \right)^{f(s_2, \frac{4}{3}) / \frac{2(m-2)}{(m-2)+1}} \\ & \leq \left( \sum_{i_{m-1}, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \right)^{f(\frac{4}{3}, s_2)}. \end{aligned}$$

Now we need to estimate the two factors above. We will write  $dt := dt_1 \dots dt_{m-2}$ .

For each  $i_{m-1}, i_m$  fixed, we have (from Theorem 1.3),

$$\begin{aligned} & \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \leq \\ & \leq \left( A_{2, \frac{4}{3}}^{m-2} \right)^{4/3} \int_{[0,1]^{m-2}} \left| \sum_{i_1, \dots, i_{m-2}=1}^N r_{i_1}(t_1) \dots r_{i_{m-2}}(t_{m-2}) U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{4}{3}} dt \\ & = \left( A_{2, \frac{4}{3}}^{m-2} \right)^{4/3} \int_{[0,1]^{m-2}} \left| U \left( \sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m-2}=1}^N r_{i_{m-2}}(t_{m-2}) x_{i_{m-2}}^{(m-2)}, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{\frac{4}{3}} dt. \end{aligned}$$

Summing over all  $i_{m-1}, i_m = 1, \dots, N$  we obtain

$$\begin{aligned} & \sum_{i_{m-1}, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \leq \\ & \left( A_{2, \frac{4}{3}}^{m-2} \right)^{4/3} \int_{[0,1]^{m-2}} \sum_{i_{m-1}, i_m=1}^N \left| U \left( \sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m-2}=1}^N r_{i_{m-2}}(t_{m-2}) x_{i_{m-2}}^{(m-2)}, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{\frac{4}{3}} dt. \end{aligned}$$

Using the case  $m = 2$  we thus have

$$\begin{aligned}
& \sum_{i_{m-1}, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \leq \\
& \leq \left( A_{2, \frac{4}{3}}^{m-2} \right)^{\frac{4}{3}} \int_{[0,1]^{m-2}} \left\| U \left( \sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m-2}=1}^N r_{i_{m-2}}(t_{m-2}) x_{i_{m-2}}^{(m-2)}, \dots \right) \right\|_{\pi(\frac{4}{3};1)}^{\frac{4}{3}} dt \\
& \leq \left( A_{2, \frac{4}{3}}^{m-2} \right)^{\frac{4}{3}} \int_{[0,1]^{m-2}} \left( \|U\| \sqrt{2} \right)^{\frac{4}{3}} dt \\
& = \left( A_{2, \frac{4}{3}}^{m-2} \right)^{\frac{4}{3}} \left( \|U\| \sqrt{2} \right)^{\frac{4}{3}}.
\end{aligned}$$

Hence

$$\left( \sum_{i_{m-1}, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq A_{2, \frac{4}{3}}^{m-2} \|U\| \sqrt{2}.$$

Next we obtain the other estimate. For each  $i_1, \dots, i_{m-2}$  fixed, and  $dt := dt_{m-1} dt_m$ , we have (from Theorem 1.3):

$$\begin{aligned}
& \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_{m-1}, i_m=1}^N \right\|_2 \leq \\
& \leq A_{2, s_2}^2 \left( \int_{[0,1]^2} \left| \sum_{i_{m-1}, i_m=1}^N r_{i_{m-1}}(t_{m-1}) r_{i_m}(t_m) U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{s_2} dt \right)^{1/s_2} \\
& = A_{2, s_2}^2 \left( \int_{[0,1]^2} \left| U \left( x_{i_1}^{(1)}, \dots, x_{i_{m-2}}^{(m-2)}, \sum_{i_{m-1}=1}^N r_{i_{m-1}}(t_{m-1}) x_{i_{m-1}}^{(m-1)}, \sum_{i_m=1}^N r_{i_m}(t_m) x_{i_m}^{(m)} \right) \right|^{s_2} dt \right)^{1/s_2}.
\end{aligned}$$

Summing over all  $i_1, \dots, i_{m-2} = 1, \dots, N$  we get:

$$\begin{aligned}
& \sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_m=1}^N \right\|_2^{s_2} \leq \\
& \leq A_{2, s_2}^{2s_2} \int_{[0,1]^2} \sum_{i_1, \dots, i_{m-2}=1}^N \left| U \left( x_{i_1}^{(1)}, \dots, x_{i_{m-2}}^{(m-2)}, \sum_{i_{m-1}=1}^N r_{i_{m-1}}(t_{m-1}) x_{i_{m-1}}^{(m-1)}, \sum_{i_m=1}^N r_{i_m}(t_m) x_{i_m}^{(m)} \right) \right|^{s_2} dt.
\end{aligned}$$

We thus have, by the induction step,

$$\begin{aligned}
& \sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_m=1}^N \right\|_2^{s_2} \leq \\
& \leq (A_{2, s_2}^2)^{s_2} \leq \int_{[0,1]} \left\| U \left( \cdot, \dots, \sum_{i_{m-1}=1}^N r_{i_{m-1}}(t_{m-1}) x_{i_{m-1}}^{(m-1)}, \sum_{i_m=1}^N r_{i_m}(t_m) x_{i_m}^{(m)} \right) \right\|_{\pi(s_2;1)}^{s_2} dt \\
& \leq (A_{2, s_2}^2)^{s_2} \int_{[0,1]} C_{\mathbb{R}, m-2}^{s_2} \|U\|^{s_2} dt = (A_{2, s_2}^2)^{s_2} C_{\mathbb{R}, m-2}^{s_2} \|U\|^{s_2}
\end{aligned}$$

and so

$$\left( \sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_m=1}^N \right\|_2^{s_2} \right)^{1/s_2} \leq (A_{2, s_2}^2) C_{\mathbb{R}, m-2} \|U\|.$$

Hence, combining both estimates, we obtain

$$\left( \sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} \leq \left[ A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \|U\| \right]^{f(\frac{4}{3}, s_2)} \left[ (A_{2, s_2}^2) C_{\mathbb{R}, m-2} \|U\| \right]^{f(s_2, \frac{4}{3})}.$$

Also,

$$f\left(\frac{4}{3}, s_2\right) = \frac{4\frac{4}{3} - 2\frac{4}{3} \left(\frac{2m-4}{m-1}\right)}{4\left(\frac{4}{3} + \frac{2m-4}{m-1}\right) - 4\frac{4}{3} \left(\frac{2m-4}{m-1}\right)} = \frac{2}{m},$$

$$f\left(s_2, \frac{4}{3}\right) = \frac{4\left(\frac{2m-2}{m}\right) - 2\left(\frac{2m-2}{m}\right)}{4\left(1 + \frac{2m-2}{m}\right) - 4\left(\frac{2m-2}{m}\right)} = 1 - \frac{2}{m},$$

and, therefore

$$\begin{aligned} \left( \sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} &\leq \left[ A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \|U\| \right]^{\frac{2}{m}} \left[ (A_{2, \frac{2m-4}{m-1}}^2) C_{\mathbb{R}, m-2} \|U\| \right]^{1 - \frac{2}{m}} \\ &= \left[ A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \right]^{\frac{2}{m}} \left( A_{2, \frac{2m-4}{m-1}}^2 \right)^{1 - \frac{2}{m}} C_{\mathbb{R}, m-2}^{1 - \frac{2}{m}} \|U\| \\ &= \left[ A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \right]^{\frac{2}{m}} \left( A_{2, \frac{2m-4}{m-1}}^2 C_{\mathbb{R}, m-2} \right)^{1 - \frac{2}{m}} \|U\| \\ &= 2^{\frac{1}{m}} \left( A_{2, \frac{4}{3}} A_{2, \frac{2m-4}{m-1}} \right)^{\frac{2m-4}{m}} (C_{\mathbb{R}, m-2})^{\frac{m-2}{m}} \|U\|. \end{aligned}$$

Now let us estimate the constants  $C_{\mathbb{R}, m}$ . We know that  $B_2 = 1$  and, from [11], we also know that  $A_p = 2^{\frac{1}{2} - \frac{1}{p}}$  whenever  $p \leq 1.847$ . So, for  $2 \leq m \leq 14$  we have

$$(2.7) \quad A_{\frac{2m-4}{m-1}} = 2^{\frac{1}{2} - \frac{m-1}{2m-4}}.$$

Hence, from (1.3) and using the best constants of Khinchine's inequality from [11], we have

$$A_{2, \frac{4}{3}} \leq A_{\frac{4}{3}}^{-1} = 2^{\frac{3}{4} - \frac{1}{2}},$$

$$A_{2, \frac{2m-4}{m-1}} \leq A_{\frac{2m-4}{m-1}}^{-1} = 2^{\frac{m-1}{2m-4} - \frac{1}{2}},$$

and

$$C_{\mathbb{R}, m} \leq 2^{\frac{1}{m}} \left( \left( 2^{\frac{3}{4} - \frac{1}{2}} \right) \left( 2^{\frac{m-1}{2m-4} - \frac{1}{2}} \right) \right)^{\frac{2m-4}{m}} (C_{\mathbb{R}, m-2})^{1 - \frac{2}{m}} = 2^{\frac{m+2}{2m}} (C_{\mathbb{R}, m-2})^{1 - \frac{2}{m}}$$

obtaining that, if  $2 \leq m \leq 14$ ,

$$C_{\mathbb{R}, m} \leq 2^{\frac{m^2+6m-8}{8m}} \text{ if } m \text{ is even,}$$

$$C_{\mathbb{R}, m} \leq 2^{\frac{m^2+6m-7}{8m}} \text{ if } m \text{ is odd.}$$

In general we easily get

$$C_{\mathbb{R}, m} \leq 2^{\frac{1}{2}} \left( \frac{C_{\mathbb{R}, m-2}}{A_{\frac{2m-4}{m-1}}^2} \right)^{\frac{m-2}{m}}.$$

The numerical values of  $C_{\mathbb{R}, m}$ , for  $m > 14$ , can be easily calculated by using the exact values of  $A_{\frac{2m-4}{m-1}}$  (see [11]):

$$A_{\frac{2m-4}{m-1}} = \sqrt{2} \left( \frac{\Gamma\left(\frac{\frac{2m-4}{m-1} + 1}{2}\right)}{\sqrt{\pi}} \right)^{(m-1)/(2m-4)}.$$

□

In the below table we compare the first constants  $C_m = 2^{\frac{m-1}{2}}$  and the constants that can be derived from [9, Cor. 5.2] with the *new* constants  $C_{\mathbb{R}, m}$ :

$m$	$C_{\mathbb{R},m}$ (using (2.5) and (2.6))	Constants from [9, Cor. 5.2])	$C_m = 2^{\frac{m-1}{2}}$
3	$2^{20/24} \approx 1.782$	$2^{5/6} \approx 1.782$	$2^{2/2} = 2$
4	$2^{32/32} = 2$	$2^{\frac{18}{16}} \approx 2.18$	$2^{3/2} \approx 2.828$
5	$2^{\frac{48}{40}} \approx 2.298$	$2^{\frac{36}{20}} \approx 2.639$	$2^2 = 4$
6	$2^{\frac{64}{48}} \approx 2.520$	$2^{\frac{40}{24}} \approx 3.17$	$2^{5/2} \approx 5.656$
7	$2^{\frac{84}{56}} \approx 2.828$	$2^{\frac{54}{28}} \approx 3.807$	$2^{6/2} = 8$
8	$2^{\frac{104}{64}} \approx 3.084$	$2^{\frac{70}{32}} \approx 4.555$	$2^{7/2} \approx 11.313$
9	$2^{\frac{128}{72}} \approx 3.429$	$2^{\frac{88}{36}} \approx 5.443$	$2^{8/2} = 16$
10	$2^{\frac{152}{80}} \approx 3.732$	$2^{\frac{108}{40}} \approx 6.498$	$2^{9/2} \approx 22.627$
11	$2^{\frac{180}{88}} \approx 4.128$	$2^{\frac{130}{44}} \approx 7.752$	$2^{10/2} = 32$
12	$2^{\frac{208}{96}} \approx 4.490$	$2^{\frac{154}{48}} \approx 9.243$	$2^{11/2} \approx 45.254$
13	$2^{\frac{240}{104}} \approx 4.951$	$2^{\frac{180}{52}} \approx 11.016$	$2^{12/2} = 64$
14	$2^{\frac{272}{112}} \approx 5.383$	$2^{\frac{13}{28}} \left( \frac{2^{\frac{180}{52}}}{A_{\frac{26}{14}}} \right)^{1-\frac{1}{14}} \approx 13.126$	$2^{13/2} \approx 90.509$

In the column at the center of the previous table we have used equations (2.7) and (2.2) for  $3 \leq m \leq 13$ . In the last line of this same column ( $m = 14$ ) we have used equation (2.2) together

with the fact that that  $A_{\frac{26}{14}} = \sqrt{2} \left( \frac{\Gamma\left(\frac{\frac{26}{14}+1}{2}\right)}{\sqrt{\pi}} \right)^{14/26} \approx 0.9736$ .

**Remark 2.3.** *In this section we have actually shown that the new constants obtained present a better asymptotic behavior than the previous ones (including those derived from [9, Cor. 5.2]). Indeed, we have previously seen that*

$$C_{\mathbb{R},m} \leq 2^{\frac{1}{2}} \left( \frac{C_{\mathbb{R},m-2}}{A_{\frac{2m-4}{m-1}}^2} \right)^{\frac{m-2}{m}}.$$

As  $m \rightarrow \infty$  we know that  $A_{\frac{2m-4}{m-1}}$  increases to 1. So,

$$\limsup \frac{C_{\mathbb{R},m}}{(C_{\mathbb{R},m-2})^{\frac{m-2}{m}}} \leq 2^{\frac{1}{2}}.$$

For the original constants  $C_m = 2^{\frac{m-1}{2}}$  we have

$$\frac{C_m}{(C_{m-2})^{\frac{m-2}{m}}} = 2^{\frac{2m-3}{m}}$$

and thus

$$\lim \frac{C_m}{(C_{m-2})^{\frac{m-2}{m}}} = 4.$$

Also, for the constants from [9, Cor. 5.2], a similar calculation shows us that  $2^{\frac{1}{2}}$  is replaced by 2 in (2.3). To summarize, these new constants, although smaller than the “old ones”, have the best asymptotic behavior.

### 3. IMPROVED CONSTANTS FOR THE BOHNENBLUST-HILLE THEOREM: THE COMPLEX CASE

As in the real case, following the proof of [9, Corollary 5.2] and using the optimal values for the constants of Khinchine’s inequality (due to Haagerup) and using that  $K_G = C_{\mathbb{C},2}$  (see [1]), the following estimates can be calculated for  $C_m$ :

$$C_{\mathbb{C},2} = K_G \leq 1.4049 < \sqrt{2},$$

$$C_{\mathbb{C},m} = 2^{\frac{m-1}{2m}} \left( \frac{C_{\mathbb{C},m-1}}{A_{\frac{2m-2}{m}}} \right)^{1-\frac{1}{m}} \quad \text{for } m \geq 3,$$

In particular, if  $2 \leq m \leq 13$ ,

$$C_{\mathbb{C},m} \leq 2^{\frac{m^2+m-6}{4m}} K_G^{2/m}$$

The above estimates are much better than  $C_{\mathbb{C},m} = 2^{\frac{m-1}{2}}$  but worst than the constants  $C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  obtained by Defant and Sevilla-Peris [8]. However, our approach will provide even better constants.

The following lemma is essentially the main result from the previous section which comes from [9], although now we will obtain different constants, since we will be dealing with the complex case.

**Lemma 3.1.** *For every positive integer  $m$  and complex Banach spaces  $X_1, \dots, X_m$ ,*

$$\Pi_{(\frac{2m}{m+1};1)}(X_1, \dots, X_m; \mathbb{C}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{C}) \text{ and } \|\cdot\|_{\Pi_{(\frac{2m}{m+1};1)}} \leq C_{\mathbb{C},m} \|\cdot\|$$

with

$$C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1} \text{ for } m = 2, 3,$$

$$C_{\mathbb{C},m} \leq \frac{2^{\frac{m+2}{2m}}}{\pi^{1/m}} \left(\frac{1}{A_{\frac{2m-4}{m-1}}^2}\right)^{\frac{m-2}{m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}} \text{ for } m \geq 4.$$

In particular, if  $4 \leq m \leq 14$  we have

$$C_{\mathbb{C},m} \leq \left(\frac{1}{\pi^{1/m}}\right) 2^{\frac{m+4}{2m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}}.$$

The proof of this result is essentially in the same spirit as that of Theorem 2.2. The cases of  $C_{\mathbb{C},2}$  and  $C_{\mathbb{C},3}$  are already known and the proof is (also) done by induction, using the cases  $m-2$  and 2 in order to achieve the case  $m$ . By proceeding in this way one obtains, at the end, that

$$C_{\mathbb{C},m} \leq \frac{2^{\frac{m+2}{2m}}}{\pi^{1/m}} \left(\frac{1}{A_{\frac{2m-4}{m-1}}^2}\right)^{\frac{m-2}{m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}},$$

and for  $2 \leq m \leq 14$  we have

$$A_{\frac{2m-4}{m-1}} = 2^{\frac{1}{2} - \frac{m-1}{2m-4}} = 2^{\frac{-1}{2m-4}},$$

which leads to

$$C_{\mathbb{C},m} \leq \frac{2^{\frac{m+2}{2m}}}{\pi^{1/m}} \left(2^{\frac{1}{m-2}}\right)^{\frac{m-2}{m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}} = \left(\frac{1}{\pi^{1/m}}\right) 2^{\frac{m+4}{2m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}}$$

for  $4 \leq m \leq 14$ .

**3.1. Comparing the “first” constants.** The first constants  $D_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  from [8] are better than the constants that we have obtained in the previous lemma. However

$$\lim \frac{D_m}{(D_{m-2})^{\frac{m-2}{m}}} = \left(\frac{2}{\sqrt{\pi}}\right)^4 \approx 1.621 > \sqrt{2} = \limsup \frac{C_{\mathbb{C},m}}{(C_{\mathbb{C},m-2})^{\frac{m-2}{m}}}.$$

So, our constants are asymptotically better, and from a certain level  $m$ , they will be better than  $D_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ . We will show that this happens when  $m \geq 8$ .

Below we compare the first constants:

For the case  $m = 4$ , notice that we have

$$C_{\mathbb{C},4} \leq \frac{2}{\pi^{1/4}} (C_{\mathbb{C},2})^{\frac{1}{2}} = \frac{2}{\pi^{1/4}} \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} = \frac{2^{3/2}}{\pi^{1/2}} \approx 1.5957$$

but this constant is *worst* than  $\left(\frac{2}{\sqrt{\pi}}\right)^3 \approx 1.437$ . So, in order to improve the constants that follow, it would be better to consider  $\left(\frac{2}{\sqrt{\pi}}\right)^3$  instead of  $\frac{2}{\pi^{1/4}} (C_{\mathbb{C},2})^{\frac{1}{2}}$  for the value of  $C_{\mathbb{C},4}$ .



Similarly, for the cases  $5 \leq m \leq 7$  we also have that  $C_{\mathbb{C},m}$  is slightly *worst* than  $(2/\sqrt{\pi})^{m-1}$  but, for  $m \geq 8$  our constants are *better* than the *old* ones.

For instance, for  $m = 8$  the situation is different. We have

$$C_{\mathbb{C},8} \leq \left(\frac{1}{\pi^{1/8}}\right) 2^{\frac{12}{16}} (C_{\mathbb{C},6})^{\frac{6}{8}} = \left(\frac{1}{\pi^{1/8}}\right) 2^{\frac{12}{16}} \left(\left(\frac{2}{\sqrt{\pi}}\right)^5\right)^{\frac{6}{8}} = \frac{2^{36/8}}{\pi^2} \approx 2.293$$

and now this constant is better than  $\left(\frac{2}{\sqrt{\pi}}\right)^7 \approx 2.329$ . Also, as we announced, for  $m > 8$  our constants are better.

In the next section we state the previous lemma using the previous information

**3.2. Comparing the “remaining” constants ( $m > 8$ ).** Now it is time to state the last lemma adding the *better* constants:

**Theorem 3.2.** *For every positive integer  $m$  and every complex Banach spaces  $X_1, \dots, X_m$ ,*

$$\Pi_{(\frac{2m}{m+1}, 1)}(X_1, \dots, X_m; \mathbb{C}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{C}) \text{ and } \|\cdot\|_{\pi(\frac{2m}{m+1}, 1)} \leq C_{\mathbb{C},m} \|\cdot\|$$

with

$$C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1} \text{ for } m = 2, 3, 4, 5, 6, 7,$$

$$C_{\mathbb{C},m} \leq \frac{2^{\frac{m+2}{2m}}}{\pi^{1/m}} \left(\frac{1}{A_{\frac{2m-4}{m-1}}^2}\right)^{\frac{m-2}{m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}} \text{ for } m \geq 8.$$

In particular, for  $8 \leq m \leq 14$  we have

$$C_{\mathbb{C},m} \leq \left(\frac{1}{\pi^{1/m}}\right) 2^{\frac{m+4}{2m}} (C_{\mathbb{C},m-2})^{\frac{m-2}{m}}.$$

Keeping in mind that for  $m > 14$ , the evaluation of the precise values of  $A_p$  need the use of Gamma function (see [11])

$$A_p = \sqrt{2} \left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\right)^{1/p},$$

we have that (assuming some slight rounding error, for high values of  $m$ , due to computer calculus):

$m$	New Constants	$C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ (from [8])	$C_m = 2^{\frac{m-1}{2}}$
8	$\approx 2.293$	$\approx 2.329$	$\approx 11.313$
9	$\approx 2.552$	$\approx 2.628$	16
10	$\approx 2.814$	$\approx 2.965$	$\approx 22.627$
11	$\approx 3.059$	$\approx 3.346$	32
12	$\approx 3.417$	$\approx 3.775$	$\approx 45.425$
13	$\approx 3.711$	$\approx 4.260$	64
14	$\approx 4.125$	$\approx 4.807$	$\approx 90.509$
15	$\approx 4.479$	$\approx 5.425$	128
16	$\approx 4.963$	$\approx 6.121$	$\approx 181.019$
50	$\approx 100$	$\approx 372$	$\approx 23,726,566$
100	$\approx 7,761$	$\approx 155,973$	$\approx 7.96131459 \cdot 10^{14}$

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,  
 FACULTAD DE CIENCIAS MATEMÁTICAS,  
 PLAZA DE CIENCIAS 3,  
 UNIVERSIDAD COMPLUTENSE DE MADRID,  
 MADRID, 28040, SPAIN.  
*E-mail address:* jseoane@mat.ucm.es

DEPARTAMENTO DE MATEMÁTICA,  
 UNIVERSIDADE FEDERAL DA PARAÍBA,  
 58.051-900 - JOÃO PESSOA, BRAZIL.  
*E-mail address:* dmpellegrino@gmail.com

# NEW UPPER BOUNDS FOR THE CONSTANTS IN THE BOHNENBLUST-HILLE INEQUALITY

DANIEL PELLEGRINO\* AND JUAN B. SEOANE-SEPÚLVEDA\*\*

ABSTRACT. A classical inequality due to Bohnenblust and Hille states that for every positive integer  $m$  there is a constant  $C_m > 0$  so that

$$\left( \sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|U\|$$

for every positive integer  $N$  and every  $m$ -linear mapping  $U : \ell_\infty^N \times \dots \times \ell_\infty^N \rightarrow \mathbb{C}$ , where  $C_m = m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$ . The value of  $C_m$  was improved to  $C_m = 2^{\frac{m-1}{2}}$  by S. Kaijser and more recently H. Quéffelec and A. Defant and P. Sevilla-Peris remarked that  $C_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  also works. The Bohnenblust–Hille inequality also holds for real Banach spaces with the constants  $C_m = 2^{\frac{m-1}{2}}$ . In this note we show that a recent new proof of the Bohnenblust–Hille inequality (due to Defant, Popa and Schwarting) provides, in fact, quite better estimates for  $C_m$  for all values of  $m \in \mathbb{N}$ . In particular, we will also show that, for real scalars, if  $m$  is even with  $2 \leq m \leq 24$ , then

$$C_{\mathbb{R},m} = 2^{\frac{1}{2}} C_{\mathbb{R},m/2}.$$

We will mainly work on a paper by Defant, Popa and Schwarting, giving some remarks about their work and explaining how to, numerically, improve the previously mentioned constants.

## 1. PRELIMINARIES AND BACKGROUND

In 1930, Littlewood proved that

$$\left( \sum_{i,j=1}^N |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \|U\|$$

for every bilinear form  $U : \ell_\infty^N \times \ell_\infty^N \rightarrow \mathbb{C}$  and every positive integer  $N$ . This is the well-known Littlewood’s 4/3 inequality [16].

One year later, in 1931, Bohnenblust and Hille ([2]) improved this result to multilinear forms (see also [8, 9] for recent approaches). More precisely, the Bohnenblust–Hille inequality asserts that for every positive integer  $m$  there is a  $C_m > 0$  so that

$$\left( \sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|U\|$$

for every  $m$ -linear mapping  $U : \ell_\infty^N \times \dots \times \ell_\infty^N \rightarrow \mathbb{C}$  and every positive integer  $N$  (for polynomial versions of Bohnenblust–Hille inequality we refer to [10]). The original upper estimate for  $C_m$  is  $m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$ , but several improvements have been obtained since then. For instance, as an illustration for the complex case we compare, below, and for some values of  $m$ , the original constants with the improvements obtained by S. Kaijser [14] and H. Quéffelec [18], Defant, P. Sevilla-Peris [8]:

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$m$	$C_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1} ([18, 8], 1995)$	$C_m = 2^{\frac{m-1}{2}} ([14], 1978)$	$C_m = m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}} ([2], 1931)$
3	$\approx 1.273$	2	$\approx 4.160$
4	$\approx 1.437$	$\approx 2.828$	$\approx 6.726$
5	$\approx 1.621$	4	$\approx 10.506$
6	$\approx 1.829$	$\approx 5.657$	$\approx 16.088$
7	$\approx 2.064$	8	$\approx 24.322$
8	$\approx 2.330$	$\approx 11.314$	$\approx 36.442$
9	$\approx 2.628$	16	$\approx 54.232$
10	$\approx 2.965$	$\approx 22.627$	$\approx 80.283$
15	$\approx 5.425$	128	$\approx 542.574$
50	$\approx 372$	$\approx 23,726,566$	$\approx 174,465,512$
100	$\approx 155,973$	$\approx 7.96131459 \cdot 10^{14}$	$\approx 8.14675743 \cdot 10^{15}$

The Bohnenblust–Hille inequality also holds for real Banach spaces but sharper estimates for  $C_m$ , in this case, seem to be  $C_m = 2^{\frac{m-1}{2}}$ .

The aim of this paper is show that improved values for  $C_m$  are essentially contained in [9] for both real and complex cases.

The (complex and real) Bohnenblust–Hille inequality can be re-written in the context of multiple summing multilinear operators, as we will see next. Multiple summing multilinear mappings between Banach spaces is a recent, very important and useful nonlinear generalization of the concept of absolutely summing linear operators. This class was introduced, independently, by Matos [17] (under the terminology fully summing multilinear mappings) and Bombal, Pérez-García and Villanueva [3]. The interested reader can also refer to [4, 5] for other Bohnenblust–Hille type results.

Throughout this paper  $X_1, \dots, X_m$  and  $Y$  will stand for Banach spaces over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $X'$  stands for the dual of  $X$ . By  $\mathcal{L}(X_1, \dots, X_m; Y)$  we denote the Banach space of all continuous  $m$ -linear mappings from  $X_1 \times \dots \times X_m$  to  $Y$  with the usual sup norm. For  $x_1, \dots, x_n$  in  $X$ , let

$$\|(x_j)_{j=1}^n\|_{w,1} := \sup\{\|(\varphi(x_j))_{j=1}^n\|_1 : \varphi \in X', \|\varphi\| \leq 1\}.$$

If  $1 \leq p < \infty$ , an  $m$ -linear mapping  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(p; 1)$ -summing (denoted  $\Pi_{(p;1)}(X_1, \dots, X_m; Y)$ ) if there exists a constant  $L_m \geq 0$  such that

$$(1.1) \quad \left( \sum_{j_1, \dots, j_m=1}^N \|U(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)})\|^p \right)^{\frac{1}{p}} \leq L_m \prod_{k=1}^m \left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1}$$

for every  $N \in \mathbb{N}$  and any  $x_{j_k}^{(k)} \in X_k$ ,  $j_k = 1, \dots, N$ ,  $k = 1, \dots, m$ . The infimum of the constants satisfying (1.1) is denoted by  $\|U\|_{\Pi_{(p;1)}}$ . For  $m = 1$  we have the classical concept of absolutely  $(p; 1)$ -summing operators (see, e.g. [7, 11]).

A simple reformulation of the Bohnenblust–Hille inequality asserts that every continuous  $m$ -linear form  $T : X_1 \times \dots \times X_m \rightarrow \mathbb{K}$  is multiple  $(\frac{2m}{m+1}; 1)$ -summing with  $L_m = C_m = 2^{\frac{m-1}{2}}$  (or  $L_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  for the complex case, using the estimates of Defant and Sevilla-Peris, [8]), although in the real case the best known constants seem to be  $C_m = 2^{\frac{m-1}{2}}$ .

The main goal of this paper is to calculate *better* constants for the Bohnenblust–Hille inequality in the real and complex case (which are derived from [9]). For this task we will explore the proof of a general vector-valued version of Bohnenblust–Hille inequality ([9, Theorem 5.1]).

Let us recall some results that we will need in this note. The first one is a well-known inequality due to Khinchine (see [11]):

**Theorem 1.1** (Khinchine’s inequality). *For all  $0 < p < \infty$ , there exist constants  $A_p$  and  $B_p$  such that*

$$(1.2) \quad A_p \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

for every positive integer  $N$  and scalars  $a_1, \dots, a_n$  (here,  $r_n$  denotes the  $n$ -th Rademacher function).

Above, it is clear that  $B_2 = 1$ . From (1.2) it follows that

$$(1.3) \quad \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p A_r^{-1} \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^r dt \right)^{\frac{1}{r}}$$

and the product of the constants  $B_p A_r^{-1}$  will appear later on in Theorem 1.3. The notation  $A_p$  and  $B_p$  will be kept along the paper. Next, let us recall a variation of an inequality due to Blei (see [9, Lemma 3.1]).

**Theorem 1.2** (Blei, Defant et al.). *Let  $A$  and  $B$  be two finite non-void index sets, and  $(a_{ij})_{(i,j) \in A \times B}$  a scalar matrix with positive entries, and denote its columns by  $\alpha_j = (a_{ij})_{i \in A}$  and its rows by  $\beta_i = (a_{ij})_{j \in B}$ . Then, for  $q, s_1, s_2 \geq 1$  with  $q > \max(s_1, s_2)$  we have*

$$\left( \sum_{(i,j) \in A \times B} a_{ij}^{w(s_1, s_2)} \right)^{\frac{1}{w(s_1, s_2)}} \leq \left( \sum_{i \in A} \|\beta_i\|_q^{s_1} \right)^{\frac{f(s_1, s_2)}{s_1}} \left( \sum_{j \in B} \|\alpha_j\|_q^{s_2} \right)^{\frac{f(s_2, s_1)}{s_2}},$$

with

$$w : [1, q]^2 \rightarrow [0, \infty), \quad w(x, y) := \frac{q^2(x+y) - 2qxy}{q^2 - xy},$$

$$f : [1, q]^2 \rightarrow [0, \infty), \quad f(x, y) := \frac{q^2x - qxy}{q^2(x+y) - 2qxy}.$$

The following theorem is a particular case of [9, Lemma 2.2] for  $Y = \mathbb{K}$  using that the cotype 2 constant of  $\mathbb{K}$  is 1, i.e.,  $C_2(\mathbb{K}) = 1$  (following the notation from [9]):

**Theorem 1.3** (Defant et al.). *Let  $1 \leq r \leq 2$ , and let  $(y_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^N$  be a matrix in  $\mathbb{K}$ . Then*

$$\left( \sum_{i_1, \dots, i_m=1}^N |y_{i_1, \dots, i_m}|^2 \right)^{1/2} \leq (A_{2,r})^m \left( \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^N r_{i_1}(t_1) \dots r_{i_m}(t_m) y_{i_1 \dots i_m} \right|^r dt_1 \dots dt_m \right)^{1/r},$$

where

$$A_{2,r} \leq A_r^{-1} B_2 = A_r^{-1} \text{ (since } B_2 = 1 \text{)}.$$

The meaning of  $A_{2,r}$ ,  $w$  and  $f$  from the above theorems will also be kept in the next section and  $K_G$  will denote the complex Grothendieck constant. Also, and throughout the paper,  $C_{\mathbb{K},m}$  will denote our estimates on the constants for the real or complex case ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  respectively).

## 2. IMPROVED CONSTANTS FOR THE BOHNENBLUST-HILLE THEOREM: THE REAL CASE

From the proof of [9, Corollary 5.2] and using the optimal values for the constants of Khinchine's inequality (due to Haagerup), the following estimates can be calculated for  $C_m$ :

$$(2.1) \quad C_{\mathbb{R},2} = \sqrt{2},$$

$$(2.2) \quad C_{\mathbb{R},m} = 2^{\frac{m-1}{2m}} \left( \frac{C_{\mathbb{R},m-1}}{A_{\frac{2m-2}{m}}} \right)^{1-\frac{1}{m}} \text{ for } m \geq 3.$$

In particular, if  $2 \leq m \leq 13$ ,

$$(2.3) \quad C_{\mathbb{R},m} = 2^{\frac{m^2+m-2}{4m}}.$$

If we change a little bit the induction process by obtaining the case  $m$  from the cases  $m-2$  and 2 we obtain even smaller constants. For example,

$$(2.4) \quad C_{\mathbb{R},m} = 2^{\frac{1}{2}} \left( \frac{C_{\mathbb{R},m-2}}{A_{\frac{2m-4}{m-1}}} \right)^{\frac{m-2}{m}} \text{ for } m > 3.$$

In particular, if  $2 \leq m \leq 14$  a careful calculation gives us

$$(2.5) \quad C_{\mathbb{R},m} = 2^{\frac{m^2+6m-8}{8m}} \text{ if } m \text{ is even, and}$$

$$(2.6) \quad C_{\mathbb{R},m} = 2^{\frac{m^2+6m-7}{8m}} \text{ if } m \text{ is odd.}$$

However, in the next theorem a different induction approach leads us to even smaller (and, thus, sharper) constants.

**Remark 2.1.** *It is worth mentioning that the values for the above constants are not explicitly calculated in [9] but, of course, are derived from [9]. Since our procedure below will improve the constants, we will not give much detail on the above estimates.*

The paper [9] provides, in fact, a family of constants  $C_m$  for Bohnenblust–Hille inequality. In this sense the following result is essentially contained in [9]. However, for the sake of completeness we prefer to sketch the proof from [9]. Our particular approach is chosen to obtain sharper constants:

**Theorem 2.2.** *For every positive integer  $m$  and real Banach spaces  $X_1, \dots, X_m$ ,*

$$\Pi_{(\frac{2m}{m+1};1)}(X_1, \dots, X_m; \mathbb{R}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{R}) \text{ and } \|\cdot\|_{\Pi_{(\frac{2m}{m+1};1)}} \leq C_{\mathbb{R},m} \|\cdot\|$$

with

$$C_{\mathbb{R},2} = 2^{\frac{1}{2}} \text{ and } C_{\mathbb{R},3} = 2^{\frac{5}{6}},$$

$$C_{\mathbb{R},m} = \frac{C_{\mathbb{R},m/2}}{A_{\frac{m/2}{m+2}}}$$

for  $m$  even and

$$C_{\mathbb{R},m} = \left( \frac{C_{\mathbb{R}, \frac{m-1}{2}}}{A_{\frac{\frac{m-1}{2}}{m+1}}} \right)^{f(\frac{2m-2}{m+1}, \frac{2m+2}{m+3})} \cdot \left( \frac{C_{\mathbb{R}, \frac{m+1}{2}}}{A_{\frac{\frac{m+1}{2}}{m+3}}} \right)^{f(\frac{2m+2}{m+3}, \frac{2m-2}{m+1})}$$

for  $m$  odd. In particular,

$$C_{\mathbb{R},m} = 2^{\frac{1}{2}} C_{\mathbb{R},m/2}$$

if  $m$  is even and  $2 \leq m \leq 24$ .

*Proof.* We start with the case  $m$  even. The cases  $m = 2$  (Littlewood's 4/3-inequality) and  $m = 3$  are known. We proceed by induction, obtaining the case  $m$  as a combination of the cases  $m/2$  and  $m/2$  (instead of 1 and  $m - 1$  as in [9, Corollary 5.2]).

Suppose that the result is true for  $m/2$  and let us prove for  $m$ . Let  $U \in \mathcal{L}(X_1, \dots, X_m; \mathbb{R})$  and  $N$  be a positive integer. For each  $1 \leq k \leq m$  consider  $x_1^{(k)}, \dots, x_N^{(k)} \in X_k$  so that  $\left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1} \leq 1$ ,  $k = 1, \dots, m$ .

Consider, following the notation of Theorem 1.2,

$$q = 2, \quad s_1 = s_2 = \frac{2 \cdot (\frac{m}{2})}{\frac{m}{2} + 1} = \frac{2m}{m+2}.$$

Thus,

$$w(s_1, s_2) = \frac{2m}{m+1}$$

and, from Theorem 1.2, we have

$$\begin{aligned} & \left( \sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} \leq \\ & \leq \left( \sum_{i_1, \dots, i_{m/2}=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_{(m/2)+1}, \dots, i_m=1}^N \right\|_2^{s_2} \right)^{f(s_2, s_1)/s_2} \\ & \left( \sum_{i_{(m/2)+1}, \dots, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m/2}=1}^N \right\|_2^{s_1} \right)^{f(s_1, s_2)/s_1}. \end{aligned}$$

Now we need to estimate the two factors above. We will write  $dt := dt_1 \dots dt_{m/2}$ . For each  $i_{(m/2)+1}, \dots, i_m$  fixed, we have (from Theorem 1.3),

$$\begin{aligned} & \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m/2}=1}^N \right\|_2^{s_1} \leq \\ & \leq \left( A_{2, s_1}^{m/2} \right)^{s_1} \int_{[0,1]^{m/2}} \left| \sum_{i_1, \dots, i_{m/2}=1}^N r_{i_1}(t_1) \dots r_{i_{m/2}}(t_{m/2}) U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{s_1} dt \\ & = \left( A_{2, s_1}^{m/2} \right)^{s_1} \int_{[0,1]^{m/2}} \left| U \left( \sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m/2}=1}^N r_{i_{m/2}}(t_{m/2}) x_{i_{m/2}}^{(m/2)}, x_{i_{(m/2)+1}}^{(m/2)+1}, \dots, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{s_1} dt. \end{aligned}$$

Summing over all  $i_{(m/2)+1}, \dots, i_m = 1, \dots, N$  we obtain

$$\begin{aligned} & \sum_{i_{(m/2)+1}, \dots, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m/2}=1}^N \right\|_2^{s_1} \leq \\ & \left( A_{2, s_1}^{m/2} \right)^{s_1} \int_{[0,1]^{m/2}} \sum_{i_{(m/2)+1}, \dots, i_m=1}^N \left| U \left( \sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m/2}=1}^N r_{i_{m/2}}(t_{m/2}) x_{i_{m/2}}^{(m/2)}, \dots, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{s_1} dt. \end{aligned}$$

Using the case  $m/2$  we thus have

$$\begin{aligned} & \sum_{i_{(m/2)+1}, \dots, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m/2}=1}^N \right\|_2^{s_1} \leq \\ & \leq \left( A_{2, s_1}^{m/2} \right)^{s_1} \int_{[0,1]^{m/2}} \left\| U \left( \sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m/2}=1}^N r_{i_{m/2}}(t_{m/2}) x_{i_{m/2}}^{(m/2)}, \dots \right) \right\|_{\pi(s_1; 1)}^{s_1} dt \\ & \leq \left( A_{2, s_1}^{m/2} \right)^{s_1} (\|U\| C_{\mathbb{R}, m/2})^{s_1}. \end{aligned}$$

Hence

$$\left( \sum_{i_{(m/2)+1}, \dots, i_m=1}^N \left\| \left( U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m/2}=1}^N \right\|_2^{s_1} \right)^{\frac{1}{s_1}} \leq A_{2, s_1}^{m/2} \|U\| C_{\mathbb{R}, m/2}.$$

The other estimate is exactly the same. Hence, combining both estimates, we obtain

$$\left( \sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} \leq \left[ A_{2, s_1}^{m/2} \|U\| C_{\mathbb{R}, m/2} \right]^{1/2} \left[ A_{2, s_1}^{m/2} \|U\| C_{\mathbb{R}, m/2} \right]^{1/2}.$$

and

$$\begin{aligned} \left( \sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} &\leq A_{2, s_1}^{m/2} \|U\| C_{\mathbb{R}, m/2} \\ &= A_{2, \frac{2m}{m+2}}^{m/2} \|U\| C_{\mathbb{R}, m/2}. \end{aligned}$$

Hence

$$C_{\mathbb{R}, m} = A_{2, \frac{2m}{m+2}}^{m/2} C_{\mathbb{R}, m/2}.$$

Now let us estimate the constants  $C_{\mathbb{R}, m}$ . We know that  $B_2 = 1$  and, from [12], we also know that  $A_p = 2^{\frac{1}{2} - \frac{1}{p}}$  whenever  $p \leq 1.847$ . So, for  $2 \leq m \leq 24$  we have

$$(2.7) \quad A_{\frac{2m}{m+2}} = 2^{\frac{1}{2} - \frac{m+2}{2m}} = 2^{-\frac{1}{m}}.$$

Hence, from (1.3) and using the best constants of Khinchine's inequality from [12], we have

$$A_{2, \frac{2m}{m+2}} \leq A_{\frac{2m}{m+2}}^{-1} = 2^{\frac{1}{m}}$$

and

$$C_{\mathbb{R}, m} \leq \left( 2^{\frac{1}{m}} \right)^{m/2} C_{\mathbb{R}, m/2} = 2^{\frac{1}{2}} C_{\mathbb{R}, m/2}$$

for  $m$  even,  $2 \leq m \leq 24$ .

The numerical values of  $C_{\mathbb{R}, m}$ , for  $m > 24$ , can be easily calculated by using the exact values of  $A_{\frac{2m}{m+2}}$  (see [12]):

$$A_{\frac{2m}{m+2}} = \sqrt{2} \left( \frac{\Gamma\left(\frac{\frac{2m}{m+2} + 1}{2}\right)}{\sqrt{\pi}} \right)^{(m+2)/2m}.$$

For the case  $m$  odd we proceed by induction, but the case  $m$  is obtained as a combination of the cases  $\frac{m-1}{2}$  with  $\frac{m+1}{2}$  instead of 1 and  $m-1$  as in [9, Corollary 5.2].

Consider, in the notation of Theorem 1.2,

$$q = 2, \quad s_1 = \frac{2\left(\frac{m-1}{2}\right)}{\frac{m-1}{2} + 1} = \frac{2m-2}{m+1} \quad \text{and} \quad s_2 = \frac{2\left(\frac{m+1}{2}\right)}{\frac{m+1}{2} + 1} = \frac{2m+2}{m+3}.$$

Thus,

$$w(s_1, s_2) = \frac{2m}{m+1}$$

and a similar proof gives us

$$\begin{aligned} C_{\mathbb{R}, m} &= \left( A_{2, \frac{2m-2}{m+1}} C_{\mathbb{R}, \frac{m-1}{2}} \right)^{f\left(\frac{2m-2}{m+1}, \frac{2m+2}{m+3}\right)} \cdot \left( A_{2, \frac{2m+2}{m+3}} C_{\mathbb{R}, \frac{m+1}{2}} \right)^{f\left(\frac{2m+2}{m+3}, \frac{2m-2}{m+1}\right)} \\ &\leq \left( \frac{C_{\mathbb{R}, \frac{m-1}{2}}}{A_{\frac{2m-2}{m+1}}} \right)^{f\left(\frac{2m-2}{m+1}, \frac{2m+2}{m+3}\right)} \cdot \left( \frac{C_{\mathbb{R}, \frac{m+1}{2}}}{A_{\frac{2m+2}{m+3}}} \right)^{f\left(\frac{2m+2}{m+3}, \frac{2m-2}{m+1}\right)} \end{aligned}$$

□

In the below table we compare the first constants  $C_m = 2^{\frac{m-1}{2}}$  and the constants that can be derived from [9, Cor. 5.2] with the *new* constants  $C_{\mathbb{R}, m}$ :



$m$	New constants - $C_{\mathbb{R},m}$	$C_m$ derived from ([9, Corollary 5.2], 2010)	$C_m = 2^{\frac{m-1}{2}}$ ([14], 1978)
3	$\approx 1.782$	$2^{5/6} \approx 1.782$	$2^{2/2} = 2$
4	2	$2^{\frac{18}{16}} \approx 2.18$	$2^{3/2} \approx 2.828$
5	$\approx 2.298$	$2^{\frac{28}{20}} \approx 2.639$	$2^2 = 4$
6	$\approx 2.520$	$2^{\frac{40}{24}} \approx 3.17$	$2^{5/2} \approx 5.656$
7	$\approx 2.6918$	$2^{\frac{54}{28}} \approx 3.807$	$2^{6/2} = 8$
8	$\approx 2.8284$	$2^{\frac{70}{32}} \approx 4.555$	$2^{7/2} \approx 11.313$
9	$\approx 3.055$	$2^{\frac{88}{36}} \approx 5.443$	$2^{8/2} = 16$
10	$\approx 3.249$	$2^{\frac{108}{40}} \approx 6.498$	$2^{9/2} \approx 22.627$
11	$\approx 3.4174$	$2^{\frac{130}{44}} \approx 7.752$	$2^{10/2} = 32$
12	$\approx 3.563$	$2^{\frac{154}{48}} \approx 9.243$	$2^{11/2} \approx 45.254$

### 3. IMPROVED CONSTANTS FOR THE BOHNENBLUST–HILLE THEOREM: THE COMPLEX CASE

As in the real case, following the proof of [9, Corollary 5.2] and using the optimal values for the constants of Khinchine's inequality (due to Haagerup) and using that  $K_G = C_{\mathbb{C},2}$  (see [1]), the following estimates can be calculated for  $C_m$ :

$$C_{\mathbb{C},2} = K_G \leq 1.4049 < \sqrt{2},$$

$$C_{\mathbb{C},m} = 2^{\frac{m-1}{2m}} \left( \frac{C_{\mathbb{C},m-1}}{A_{\frac{2m-2}{m}}} \right)^{1-\frac{1}{m}} \quad \text{for } m \geq 3,$$

In particular, if  $2 \leq m \leq 13$ ,

$$C_{\mathbb{C},m} \leq 2^{\frac{m^2+m-6}{4m}} K_G^{2/m}.$$

The above estimates improve the values  $C_{\mathbb{C},m} = 2^{\frac{m-1}{2}}$  but are worst than the constants  $C_{\mathbb{C},m} = \left( \frac{2}{\sqrt{\pi}} \right)^{m-1}$  obtained by Defant and Sevilla-Peris [8]. However, our approach will provide even better constants.

The following lemma is essentially the main result from the previous section which comes from [9], although now we will obtain different constants, since we will be dealing with the complex case.

**Lemma 3.1.** *For every positive integer  $m$  and complex Banach spaces  $X_1, \dots, X_m$ ,*

$$\Pi_{(\frac{2m}{m+1}, 1)}(X_1, \dots, X_m; \mathbb{C}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{C}) \quad \text{and} \quad \|\cdot\|_{\Pi_{(\frac{2m}{m+1}, 1)}} \leq C_{\mathbb{C},m} \|\cdot\|$$

with

$$C_{\mathbb{C},m} = \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} \quad \text{for } m = 2, 3,$$

$$C_{\mathbb{C},m} = \frac{C_{\mathbb{C},m/2}}{A_{\frac{2m}{m+2}}^{m/2}}$$

for  $m$  even and

$$C_{\mathbb{C},m} = \left( \frac{C_{\mathbb{C}, \frac{m-1}{2}}}{A_{\frac{2}{m+1}}^{\frac{m-1}{2}}} \right)^{f(\frac{2m-2}{m+1}, \frac{2m+2}{m+3})} \left( \frac{C_{\mathbb{C}, \frac{m+1}{2}}}{A_{\frac{2}{m+3}}^{\frac{m+1}{2}}} \right)^{f(\frac{2m+2}{m+3}, \frac{2m-2}{m+1})}$$

for  $m$  odd.

The proof of this result is essentially in the same spirit as that of Theorem 2.2. The cases of  $C_{\mathbb{C},2}$  and  $C_{\mathbb{C},3}$  are already known and the proof is (also) done by induction.

**3.1. Comparing the “first” constants.** The first constants  $D_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  from [8] are better than the constants that we have obtained in the previous lemma. However our constants present a smaller asymptotical growth and, from a certain level  $m$  on, they are better than  $D_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ . As we see next, this occurs when  $m \geq 7$ .

Here below we compare the first constants. For the case  $m = 4$ , notice that we have

$$C_{\mathbb{C},4} = 2^{1/2} \cdot C_{\mathbb{C},2} = 2^{1/2} \cdot \left(\frac{2}{\sqrt{\pi}}\right) \approx 1.5957$$

but this constant is *worst* than  $\left(\frac{2}{\sqrt{\pi}}\right)^3 \approx 1.437$ . So, in order to improve the constants that follow, it would be better to consider  $\left(\frac{2}{\sqrt{\pi}}\right)^3$  instead of  $2^{1/2} \cdot C_{\mathbb{C},2}$  for the value of  $C_{\mathbb{C},4}$ .

Similarly, for the cases  $5 \leq m \leq 6$  we also have that  $C_{\mathbb{C},m}$  is slightly *worst* than  $(2/\sqrt{\pi})^{m-1}$  but, for  $m = 7$  we have

$$C_{\mathbb{C},7} = \left(\frac{\left(\frac{2}{\sqrt{\pi}}\right)^2}{\left(2^{\frac{1}{2}-\frac{8}{12}}\right)^4}\right)^{3/7} \cdot \left(\frac{\left(\frac{2}{\sqrt{\pi}}\right)^3}{\left(2^{\frac{1}{2}-\frac{10}{16}}\right)^3}\right)^{4/7} = 1.9293 < \left(\frac{2}{\sqrt{\pi}}\right)^6$$

and our constants are *better* than the *old* ones. Also, as we announced, for  $m \geq 7$  our constants also improve the *old* ones.

In the next section we state the previous lemma using the information we just obtained.

**3.2. Comparing the “remaining” constants ( $m > 7$ ).** Now it is time to state the last lemma adding the *better* constants:

**Theorem 3.2.** *For every positive integer  $m$  and every complex Banach spaces  $X_1, \dots, X_m$ ,*

$$\Pi_{\left(\frac{2m}{m+1}, 1\right)}(X_1, \dots, X_m; \mathbb{C}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{C}) \text{ and } \|\cdot\|_{\pi\left(\frac{2m}{m+1}, 1\right)} \leq C_{\mathbb{C},m} \|\cdot\|$$

with

- $C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$  for  $m \in \{2, 3, 4, 5, 6\}$ ,
- $C_{\mathbb{C},m} = \frac{C_{\mathbb{C},m/2}}{A_{\frac{2m}{m+2}}^{m/2}}$  for  $m > 6$  even, and
- $C_{\mathbb{C},m} = \left(\frac{C_{\mathbb{C},\frac{m-1}{2}}}{A_{\frac{2m-2}{m+1}}^{\frac{m-1}{2}}}\right)^{f\left(\frac{2m-2}{m+1}, \frac{2m+2}{m+3}\right)} \cdot \left(\frac{C_{\mathbb{C},\frac{m+1}{2}}}{A_{\frac{2m+2}{m+3}}^{\frac{m+1}{2}}}\right)^{f\left(\frac{2m+2}{m+3}, \frac{2m-2}{m+1}\right)}$  for  $m > 5$  odd.

The following table compares these new constants with the previous ones:

$m$	New Constants - $C_{\mathbb{C},m}$	$\left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ ([18], [8], 1995)	$2^{\frac{m-1}{2}}$ ([14], 1978)	$m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$ ([2], 1931)
8	$\approx 2.031$	$\approx 2.329$	$\approx 11.313$	$\approx 36.442$
9	$\approx 2.172$	$\approx 2.628$	16	$\approx 54.232$
10	$\approx 2.292$	$\approx 2.965$	$\approx 22.627$	$\approx 80.283$
11	$\approx 2.449$	$\approx 3.346$	32	$\approx 118.354$
12	$\approx 2.587$	$\approx 3.775$	$\approx 45.425$	$\approx 173.869$
13	$\approx 2.662$	$\approx 4.260$	64	$\approx 254.680$
14	$\approx 2.728$	$\approx 4.807$	$\approx 90.509$	$\approx 372.128$
15	$\approx 2.805$	$\approx 5.425$	128	$\approx 542.574$
16	$\approx 2.873$	$\approx 6.121$	$\approx 181.019$	$\approx 789.612$

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,  
 FACULTAD DE CIENCIAS MATEMÁTICAS,  
 PLAZA DE CIENCIAS 3,  
 UNIVERSIDAD COMPLUTENSE DE MADRID,  
 MADRID, 28040, SPAIN.  
*E-mail address:* jseoane@mat.ucm.es

DEPARTAMENTO DE MATEMÁTICA,  
 UNIVERSIDADE FEDERAL DA PARAÍBA,  
 58.051-900 - JOÃO PESSOA, BRAZIL.  
*E-mail address:* pellegrino@pq.cnpq.br